Potencias de la unidad imaginaria

Sea $n \in \mathbb{Z}$. Entonces,

$$i^n = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor}\left[1 - \left(n - 2\left\lfloor \frac{n}{2} \right\rfloor\right) + \left(n - 2\left\lfloor \frac{n}{2} \right\rfloor\right) i\right]$$

Además,

$$\left\lfloor \frac{n}{2} \right\rfloor = \frac{2n - 1 + (-1)^n}{4}$$

Observación:

$$\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$$

Observación:

$$\frac{1 - (-1)^n}{2} = \begin{cases} 0, & \text{si } n \text{ es par} \\ 1, & \text{si } n \text{ es impar} \end{cases}$$

Raíces de la unidad imaginaria

Sea $n \in \mathbb{Z}^+$. Si $i = (a + bi)^n$, con $a, b \in \mathbb{R}$, entonces:

$$\begin{cases} \displaystyle\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} a^{n-2k} b^{2k} = 0 \\ \displaystyle\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} a^{n-(2k+1)} b^{2k+1} = 1 \end{cases}$$

Es decir:

- ✓ $b \neq 0$.
- ✓ $Si\ a \neq 0$, entonces,

$$a^n = \frac{1}{\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} \left(\frac{b}{a}\right)^{2k+1}}$$

En donde:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} \left(\frac{b}{a}\right)^{2k} = 0$$

$\veebar$ $a = 0$ si y solo si n es impar y $b = (-1)^{\frac{n-1}{2}}$.

Observación:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = (a + b)^n$$

Raíces de la unidad real

Sea $n \in \mathbb{Z}^+$. Si $1 = (a + bi)^n$, con $a, b \in \mathbb{R}$, entonces:

$$\begin{cases} \displaystyle\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} a^{n-2k} b^{2k} = 1 \\ \displaystyle\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} a^{n-(2k+1)} b^{2k+1} = 0 \end{cases}$$

Es decir:

- ✓ $a \neq 0$.
- ✓ Si $b \neq 0$, entonces,

$$a^n = \frac{1}{\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} \left(\frac{b}{a}\right)^{2k}}$$

En donde:

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} \left(\frac{b}{a}\right)^{2k} = 0$$

$\veebar$ $b = 0$ si y solo si $a^n = 1$.

Funciones trigonométricas para el múltiplo de un ángulo

Ahora, teniendo en cuenta que:

$$\frac{d^n \cos x}{dx^n} = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \left[\left(1 - \left(n - 2\left\lfloor \frac{n}{2} \right\rfloor\right)\right) \cos x - \left(n - 2\left\lfloor \frac{n}{2} \right\rfloor\right) \sin x \right]$$

$$\frac{d^n \sin x}{dx^n} = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \left[\left(n - 2\left\lfloor \frac{n}{2} \right\rfloor\right) \cos x + \left(1 - \left(n - 2\left\lfloor \frac{n}{2} \right\rfloor\right)\right) \sin x \right]$$

Y mediante el uso de los polinomios de Taylor centrados en cero, se obtiene:

$$\cos x = \sum_{n=0}^{\infty} (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \left(1 - \left(n - 2\left\lfloor \frac{n}{2} \right\rfloor\right)\right) \frac{x^n}{n!} \qquad \wedge \qquad \sin x = \sum_{n=0}^{\infty} (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \left(n - 2\left\lfloor \frac{n}{2} \right\rfloor\right) \frac{x^n}{n!}$$

En consecuencia,

$$\cos x + i \sin x = e^{xi}$$

Luego,

$$\cos nx + i \sin nx = (\cos x + i \sin x)^n = \sum_{k=0}^{n} \binom{n}{k} (\cos x)^{n-k} (i \sin x)^k \qquad (1)$$

Coseno del múltiplo de un ángulo

De la expresión (1):

$$\cos nx = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (-1)^k (\cos x)^{n-2k} (\sin x)^{2k}$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \left[\sum_{p=0}^{k} \binom{k}{p} (-1)^p (\cos x)^{n-2p} \right]$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\sum_{p=k}^{\lfloor n/2 \rfloor} \binom{n}{2p} \binom{p}{k} \right) (-1)^k (\cos x)^{n-2k} \qquad (2)$$

Observación:

$$\sum_{k=0}^{n} a_k \left(\sum_{p=0}^{k} b_p c_{p,k} \right) = \sum_{k=0}^{n} \left(\sum_{p=k}^{n} a_p c_{k,p} \right) b_k$$

Por otro lado,

$$\sum_{n=0}^{m} \binom{m}{n} \cos nx = \left(2 \cos \frac{x}{2} \right)^m \cos m \frac{x}{2}$$

Y,

$$\sum_{n=0}^{m} \cos nx = \frac{\cos m \frac{x}{2}}{\sin \frac{x}{2}} \sin(m+1) \frac{x}{2} \qquad (3)$$

Considerando (2),

$$\cos nx = (\cos x)^n \sum_{p=0}^{\lfloor n/2 \rfloor} \binom{n}{2p} + \sum_{k=1}^{\lfloor n/2 \rfloor} \left(\sum_{p=k}^{\lfloor n/2 \rfloor} \binom{n}{2p} \binom{p}{k} \right) (-1)^k (\cos x)^{n-2k}$$

Por consiguiente,

$$(\cos x)^n = \frac{1}{2^{n-1}} \left[\cos nx - \sum_{k=1}^{\lfloor n/2 \rfloor} \left(\sum_{p=k}^{\lfloor n/2 \rfloor} \binom{n}{2p} \binom{p}{k} \right) (-1)^k (\cos x)^{n-2k} \right]$$

Ahora, usando (3) se obtiene:

$$\sum_{m=0}^{q} (\cos mx)^n = \frac{1}{2^{n-1}} \left[\sum_{m=0}^{q} \cos nmx - \sum_{m=0}^{q} \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \left(\sum_{p=k}^{\lfloor n/2 \rfloor} \binom{n}{2p} \binom{p}{k} \right) (\cos mx)^{n-2k} \right]$$

$$= \frac{1}{2^{n-1}} \left\{ \frac{\cos qn \frac{x}{2}}{\sin n \frac{x}{2}} \sin(q+1)n \frac{x}{2} - \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \left(\sum_{p=k}^{\lfloor n/2 \rfloor} \binom{n}{2p} \binom{p}{k} \right) \left[\sum_{m=0}^{q} (\cos mx)^{n-2k} \right] \right\} \qquad (*)$$

Observación:

$$\sum_{m=0}^{q} \left(\sum_{k=1}^{n} a_k b_{m,k} \right) = \sum_{k=1}^{n} a_k \left(\sum_{m=0}^{q} b_{m,k} \right)$$

Por otro lado, si $y \in \mathbb{R}$,

$$\sum_{n=0}^{m-1} y^n \cos nx = \frac{1 - y \cos x - y^m \cos mx + y^{m+1} \cos(m-1)x}{1 - 2y \cos x + y^2}$$

Si $|y| < 1, \lim_{m \to \infty} y^m = 0$, así que:

$$\sum_{n=0}^{\infty} y^n \cos nx = \frac{1 - y\cos x}{1 - 2y\cos x + y^2}$$

Seno del múltiplo de un ángulo

De la expresión (1):

$$\sin nx = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} (\cos x)^{n-(2k+1)} (\sin x)^{2k+1}$$

En particular,

$$\sin(2n+1)x = \sum_{k=0}^{n} \binom{2n+1}{2k+1} \left[\sum_{p=k}^{n} (-1)^p \binom{n-k}{n-p} (\sin x)^{2p+1} \right]$$

$$= \sum_{k=0}^{n} \binom{2n+1}{2k+1} \left[\sum_{p=k}^{n} (-1)^p \binom{n-k}{n-p} (\sin x)^{2p+1} \right]$$

$$= \sum_{k=0}^{n} \left(\sum_{p=0}^{k} \binom{2n+1}{2p+1} \binom{n-p}{n-k} \right) (-1)^k (\sin x)^{2k+1} \qquad (4)$$

Observación:

$$\sum_{k=0}^{n} a_k \left(\sum_{p=k}^{n} b_p c_{p,k} \right) = \sum_{k=0}^{n} \left(\sum_{p=0}^{k} a_p c_{k,p} \right) b_k$$

Haciendo $x = \frac{\pi}{2n+1}$, se obtiene:

$$\sum_{k=0}^{n} \left(\sum_{p=0}^{k} \binom{2n+1}{2p+1} \binom{n-p}{n-k} \right) (-1)^k \left(\sin \frac{\pi}{2n+1} \right)^{2k} = 0$$

Para $n = 2$, se obtiene la expresión $16\left(\sin\frac{\pi}{5}\right)^4 - 20\left(\sin\frac{\pi}{5}\right)^2 + 5 = 0$. Puesto que $\sin\frac{\pi}{6} < \sin\frac{\pi}{5} < \sin\frac{\pi}{4}$, entonces,

$$\sin\frac{\pi}{5} = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$$

Por otro lado,

$$\sum_{n=0}^{m} \binom{m}{n} \sin nx = \left(2\cos\frac{x}{2}\right)^m \sin m\frac{x}{2}$$

Y,

$$\sum_{n=0}^{m} \sin nx = \frac{\sin m\frac{x}{2}}{\sin\frac{x}{2}} \sin(m+1)\frac{x}{2} \qquad (5)$$

Considerando (4),

$$\sin(2n+1)x = (-1)^n(\sin x)^{2n+1} \sum_{p=0}^{n}\binom{2n+1}{2p+1} + \sum_{k=0}^{n-1}\left(\sum_{p=0}^{k}\binom{2n+1}{2p+1}\binom{n-p}{n-k}\right)(-1)^k(\sin x)^{2k+1}$$

Por consiguiente,

$$(\sin x)^{2n+1} = \frac{(-1)^n}{(2)^{2n}}\left[\sin(2n+1)x - \sum_{k=0}^{n-1}\left(\sum_{p=0}^{k}\binom{2n+1}{2p+1}\binom{n-p}{n-k}\right)(-1)^k(\sin x)^{2k+1}\right]$$

Ahora, usando (5) se obtiene:

$$\sum_{m=0}^{q}(\sin mx)^{2n+1} = \frac{(-1)^n}{(2)^{2n}}\left\{\sum_{m=0}^{q}\sin(2n+1)mx - \sum_{m=0}^{q}\left[\sum_{k=0}^{n-1}\left(\sum_{p=0}^{k}\binom{2n+1}{2p+1}\binom{n-p}{n-k}\right)(-1)^k(\sin mx)^{2k+1}\right]\right\}$$

$$= \frac{(-1)^n}{(2)^{2n}}\left\{\frac{\sin q(2n+1)\frac{x}{2}}{\sin(2n+1)\frac{x}{2}}\sin(q+1)(2n+1)\frac{x}{2} - \sum_{k=0}^{n-1}(-1)^k\left(\sum_{p=0}^{k}\binom{2n+1}{2p+1}\binom{n-p}{n-k}\right)\left(\sum_{m=0}^{q}(\sin mx)^{2k+1}\right)\right\} \quad (**)$$

Observación:

$$\sum_{m=0}^{q}\left(\sum_{k=0}^{n}a_k b_{m,k}\right) = \sum_{k=0}^{n}a_k\left(\sum_{m=0}^{q}b_{m,k}\right) \ \wedge \ b_{0,k} = 0$$

Por otro lado, si $y \in \mathbb{R}$,

$$\sum_{n=0}^{m-1} y^n \sin nx = \frac{y\sin x - y^m \sin mx + y^{m+1}\sin(m-1)x}{1 - 2y\cos x + y^2}$$

Si $|y| < 1, \lim_{m\to\infty} y^m = 0$, así que:

$$\sum_{n=0}^{\infty} y^n \sin nx = \frac{y\sin x}{1 - 2y\cos x + y^2}$$

Tangente del múltiplo de un ángulo

Sea $n \in \mathbb{Z}^+$. Si $p + qi = (a+bi)^n$, con $a, b \in \mathbb{R}$, entonces:

$$\begin{cases} a^n \displaystyle\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{2k}\left(\frac{b}{a}\right)^{2k} = p \\ a^n \displaystyle\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-1)^k\binom{n}{2k+1}\left(\frac{b}{a}\right)^{2k+1} = q \end{cases}$$

En consecuencia,

$$\tan(nx) = \frac{\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-1)^k\binom{n}{2k+1}(\tan x)^{2k+1}}{\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{2k}(\tan x)^{2k}}$$

*Observación: Se considera el **teorema de Moivre**, para obtener las n raíces de un número complejo:*

$$[r(\cos x + i\sin x)]^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{x+2\pi k}{n} + i\sin\frac{x+2\pi k}{n}\right), \text{donde } k \in \{0,1,2,\dots,n-1\}$$

Serie finita de potencias

Sea $p \in \mathbb{Z}^+$.

$$S_{n,p} = \sum_{k=1}^{n} k^p r^k$$

Entonces,

$$S_{n,p} - rS_{n,p} = \sum_{k=1}^{n} k^p r^k - \sum_{k=1}^{n} k^p r^{k+1}$$

$$(1-r)S_{n,p} = \sum_{k=1}^{n} k^p r^k - \left(\sum_{k=1}^{n-1} k^p r^{k+1} + (n-1)^p r^n\right) = \sum_{k=1}^{n} k^p r^k - \left(\sum_{k=2}^{n} (k-1)^p r^k + (n-1)^p r^n\right)$$

$$= \left(\sum_{k=1}^{n} k^p r^k - \sum_{k=1}^{n} (k-1)^p r^k\right) - (n-1)^p r^n = \sum_{k=1}^{n} [k^p - (k-1)^p]\, r^k - (n-1)^p r^n$$

$$= \sum_{k=1}^{n} \left(\sum_{i=0}^{p} (-1)^{i+1} \binom{p}{i} k^{p-i}\right) r^k - (n-1)^p r^n = \sum_{i=1}^{p} (-1)^{i+1} \binom{p}{i} \left(\sum_{k=1}^{n} k^{p-i} r^k\right) - (n-1)^p r^n$$

Por consiguiente,

$$S_{n,p} = \frac{1}{1-r}\left[\sum_{i=1}^{p} (-1)^{i+1} \binom{p}{i} S_{n,p-i} - (n-1)^p r^n\right]$$

En donde,

$$S_{n,0} = r\,\frac{1-r^n}{1-r}$$